As the results of the numerical experiments showed, for the input 1/p the maximum deviation of the output signals of the digital model and the thermal link is observed initially (Table 1). For t > 7 sec (k = 6.7) the error does not exceed 0.1%. Increasing k decreases the output signals initially and therefore extends the initial section up to 100 sec. For t > 100 sec the error does not exceed 0.17%. For the input $1(p\sqrt{p})$ the error of the digital model is initially somewhat higher (Table 2). But for t > 15 sec (k = 6.7) it does not exceed 0.2%, For k = 67, t = 200 sec the error does not exceed 0.22%. Graphs of the output signals of the thermal link and digital model for inputs $1/(p\sqrt{p})$ and 1/p are represented in Fig. 3. For k = 6.7 the output signals of the digital model and thermal link are identical in the observation interval [10, 100 sec].

NOTATION

p, Laplace transform parameter; $\delta\Theta$, quantization step for the excess temperature Θ; $\Theta(t)$, excess temperature, $\hat{\Theta}_{out}$, is the digital signal at the input of the model, n = 0, 1, 2 ...,

$\operatorname{erfc} x = 1 - \operatorname{erf} x$, $\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x^2)\,dx$ is Gauss'error function.

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RECONSTRUCTION OF CAUSAL CHARACTERISTICS OF THE THERMAL CONDUCTIVITY PROCESS FROM THE SOLUTION OF THE COMBINED INVERSE PROBLEM

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UDC 536.24

An algorithm is suggested for solving the combined inverse problem of heat exchange on the basis of using uniqueness theorems.

Methods of inverse problems of heat exchange (IPHE), substantially enhancing the effectiveness of studies in this region, have become widely used in analyzing heat-exchange processes. Among the various IPHE formulations, one can distinguish the combined methods [1], when one seeks simultaneously causal characteristics of various types. Thus, in simulating thermal processes is heat-protection materials, during plasma deposition, heating, and a number of other cases the necessity arises of determining the thermal conductivity coefficient in the high temperature region. At the same time the low measurement accuracy does not make it possible to obtain reliable information concerning external thermal loads and internal heat sources, related, for example, to chemical reaction flow in the bulk of the material investigated. This difficulty is overcome as a result of solving the combined IPHE, consisting of determining the coefficients of the thermal conductivity equation and the heat flux density at the boundary from temperature measurements at internal points of the body. We note that in several cases the temperatures at internal points are the only reliable source of information on the thermal state of the object.

V. V. Kuibyshev Polytechnic Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 5, pp. 839-844, May, 1987. Original article submitted November 27, 1985.

0022-0841/87/5205-0617$12.50

Considering the mathematical aspects of the problem, it may be noted that uniqueness theorems play a major role in similar problems. The use of these theorems in solving IPHE substantially enhances the reliability of the reconstructed characteristics, and sharply outlines the boundaries of their uniqueness. As shown in the present study, the uniqueness theorem provides the possibility of not assuming known boundary conditions at the boundary of the body. Uniqueness theorems of the coefficients of inverse problems with data at internal points were earlier obtained under the assumption that one of the coefficients is unknown [2, 3]. In the case of two unknown coefficients it is suggested in [4] that inside the body there exists a standard sample with known properties.

In the present study we discuss an IPHE uniqueness theorem for the determination of the thermal conductivity coefficient, the bulk heat release, and the thermal flux density at one of the boundaries from temperature measurements at three internal points of the body. The theorem is proved by a modified method of studies [5-8].

Consider the following IPHE:

$$c(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + K(T)\frac{\partial T}{\partial x} + Q(T), \tag{1}$$

$$0 < x < b, \quad 0 < \tau < \tau_m,$$

$$T|_{\tau=0} = T_0, \tag{2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=b} = 0, \tag{3}$$

$$T|_{x=x_i} = f_i(\tau), \quad i = 1,\ 2,\ 3, \quad 0 < x_1 < x_2 < x_3 < x_4 = b, \tag{4}$$

$T), K(T), f_i(\tau)$, i = 1, 2, 3, are known functions. It is required to determine the vector-function $\{T,\ \lambda(T),\ Q(T),\ q(\tau)\}$, where

$$q(\tau) = -\left.\left(\lambda(T)\frac{\partial T}{\partial x}\right)\right|_{x=0} \tag{5}$$

We assume that $c(T),\ K(T),\ Q(T) \in C(-\infty,\ \infty),\ \lambda(T) \in C^1(-\infty,\ \infty),\ c(T) \geqslant \gamma_1 > 0,\ \lambda(T) \geqslant \gamma_2 > 0$. We denote $G = (0,\ b)(0,\ \tau_m),\ \overline{G} = [0,\ b]\ [0,\ \tau_m]$, where $C^{4,2}$ is a set of functions, having continuous, bounded derivatives $\partial/\partial x^m \partial \tau^n$ in $\overline{G}$, with m + 2n $\leqslant$ 4. We further take

$$T \in C^{4,2}(\overline{G}),\ \ \forall (x,\ \tau) \in G\ \ \frac{\partial T}{\partial x} < 0, \quad \frac{\partial T}{\partial \tau} > 0. \tag{6}$$

Inequality (6) implies a decrease in x and an increase in τ of the function T(x, τ). Usually (6) is obvious from physical considerations. Mathematically this can be verified by means of the maximum principle if we impose additional conditions on IPHE data [9].

It can be assumed that at low temperatures the coefficients of Eq. (1) are constant in a narrow interval. The following lemma is then valid.

LEMMA. For small number $\varepsilon > 0$ let the functions c(T), λ(T), K(T), Q(T) be constant for $T_0 < T < T_0 + \varepsilon$. The constants $\lambda_0 = \lambda(T),\ Q_0 = Q(T)$ are then uniquely determined at $(x, \tau) \in (0, b) \times (0, \delta)$ for some small number $\delta = \delta(\varepsilon)$. The number $\delta(\varepsilon)$ can be estimated from above in terms of ε.

The lemma is proved on the basis of explicit equations of the solution of the problem (1)-(3), (5) in the case of constant coefficients.

Thus, in view of the lemma it can be assumed that the functions $\xi(x) = T(x,\ \delta),\ \lambda(\xi(x)),\ Q(\xi(x))$ are known. We denote $\alpha_1 = \xi(0),\ \alpha_2 = f_1(\tau_m)$. Let $\alpha_2 > \alpha_1$. For each $z \in [\alpha_1,\ \alpha_2]$ we denote by $S_z(x)$ the function being a solution of the equation $T(x,\ S_z(x)) = z$. The function $\tau = S_z(x)$ is naturally assumed to be isothermal. We denote $D_z = \{(x,\ \tau) \in G\,|\,T(x,\ \tau) < z\}$.

THEOREM. Let the lemma conditions be satisfied, and $\forall z \in [\alpha_1,\ \alpha_2]$

$$\frac{\partial T(x_2,\ S_z(x_2))}{\partial x} \neq \frac{\partial T(x_1,\ S_z(x_1))}{\partial x} \tag{7}$$

Then no more than one vector-function can be found

$$\{T,\ \lambda(z),\ Q(z),\ q(\tau)\} \in C^{4,2}(D_{\alpha_2}) \times C^1(T_0,\ \alpha_2)\ \ C(T_0,\ \alpha_2) \times C^2(0,\ S_{\alpha_2}(0)),$$

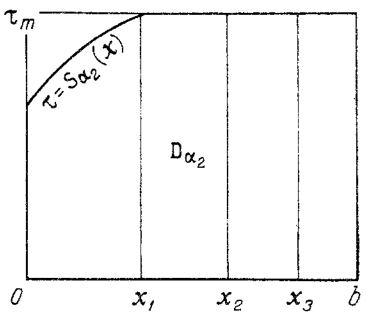

Fig. 1. The uniqueness region of the IPHE solution, defined by the theorem.

satisfying (1)-(4), (6). The region $D_{\alpha 2}$ is determined uniquely (Fig. 1).

The theorem is proved by using the modified method of studies [5-8]. A more general theorem is considered in [10].

One comment must be made concerning this theorem. It is assumed in it that the functions T, λ, Q, q are uniquely determined on the sets D_{α_2}, $[T_0, \alpha_2]$, $[T_0, \alpha_2]$, $[0, S_{\alpha_2}(0)]$, respectively. Outside the region $D_{\alpha 2}$ uniqueness no longer holds.

Below we consider the nonlinear inverse problem, in which from the unperturbed inlet temperature at three internal points of the body one reconstructs the temperature dependences of the thermal conductivity coefficient and of the bulk heat release, as well as the time dependence of the thermal flux density at the boundary.

The problem is formulated as follows. It is required to determine the functions $T(x, \tau)$, $\lambda(T)$, $Q(T)$, $q(\tau)$ from conditions (1)-(4) for each case K(T) = 0.

Since by the uniqueness theorem the parameters λ(T) and Q(T) can be determined uniquely only on the segment $[T_{min}, T_{max}]$, where $T_{min} = T_0$, $T_{max} = f_1(\tau_m)$, it is natural to partition the original problem into two problems.

In the first we have reconstruction of the functions λ(T) and Q(T) on the segment $[T_{min}, T_{max}]$ from the coefficienct solution of the inverse problem in the region $(x_1, b) \times (0, \tau_m)$ from known conditions at the boundaries (given by the temperature $f_1(\tau)$ at the point $x = x_1$ and by the vanishing thermal flux density at the point x = b) and the inlet temperature $f_2(\tau)$ and $f_3(\tau)$. A similar problem for given temperatures at both boundaries was investigated in [11] (simultaneous reconstruction of the heat conductivity and heat capacity coefficients was considered), while the presence of boundary conditions of the second kind does not lead to major changes in the algorithm.

The mathematical formualtion of the problem is:

$$c(\Theta)\frac{\partial\Theta}{\partial\tau} = \frac{\partial}{\partial x}\left(\sum_{n=0}^{N}\lambda_n\varphi_n(\Theta)\frac{\partial\Theta}{\partial x}\right) + \sum_{n=0}^{N}Q_n\varphi_n(\Theta), \tag{8}$$

$$x_1 < x < b, \quad 0 < \tau < \tau_m,$$

$$\Theta|_{\tau=0} = T_0, \tag{9}$$

$$\Theta|_{x=x_1} = f_1(\tau), \tag{10}$$

$$\left.\frac{\partial\Theta}{\partial x}\right|_{x=b} = 0. \tag{11}$$

The function λ and Q are approximated piecewise linearly on the segment $[T_{min}, T_{max}]$:

$$\lambda(\Theta) = \sum_{n=0}^{N}\lambda_n\varphi_n(\Theta), \tag{12}$$

$$Q(\Theta) = \sum_{n=0}^{N}Q_n\varphi_n(\Theta), \tag{13}$$

where N is the number of approximation sites, and $\varphi_n(\Theta)$ are pyramidal basis functions. It is required to determine the vectors $\{\lambda_n, n=\overline{0, N}\}$, $\{Q_n, n=\overline{0, N}\}$.

Considering the problem stated as extremal, we introduce into the treatment the functional

$$J(\lambda, Q) = \sum_{i=2,3} \int_0^{\tau_m} [\Theta(x_i, \tau) - f_i(\tau)]^2 d\tau. \tag{14}$$

The reconstruction of the unknown characteristics reduces in this case to minimizing (14) with restrictions on the discrepancy level.

An effective means of searching for the minimum point of the functional is the gradient descent by the iteration procedure:

$$\overset{(k)}{\lambda}_n = \overset{(k-1)}{\lambda}_n - \beta_{\lambda k} \frac{\partial \overset{(k)}{J}}{\partial \lambda_n}, \quad n=\overline{0, N}, \quad k=1, 2, \ldots, \tag{15}$$

$$\overset{(k)}{Q}_n = \overset{(k-1)}{Q}_n - \beta_{Qk} \frac{\partial \overset{(k)}{J}}{\partial Q_n}, \quad n=\overline{0, N}, \quad k=1, 2, \ldots \tag{16}$$

To calculate the gradient components of the functionals $\partial J/\partial \lambda$, $\partial J/\partial \mathbf{Q}$ we introduce into the treatment the problem conjugate to (8)-(11):

$$-c(\Theta)\frac{\partial \psi_i}{\partial \tau} = \sum_{n=0}^{N} \lambda_n \varphi_n(\Theta) \frac{\partial^2 \psi_i}{\partial x^2} + \sum_{n=0}^{N} Q_n \frac{d\varphi_n(\Theta)}{d\Theta} \psi_i,$$
$$x_i < x < x_{i+1}, \quad i=1, 2, 3, \quad 0 < \tau < \tau_m, \tag{17}$$

$$\psi_i|_{\tau=\tau_m} = 0, \quad i=1, 2, 3, \tag{18}$$

$$\psi_1|_{x=x_1} = 0, \quad \psi_{i-1}|_{x=x_i} = \psi_i|_{x=x_i}, \quad i=2, 3, \tag{19}$$

$$\sum_{n=0}^{N} \lambda_n \varphi_n(\Theta)\left(\frac{\partial \psi_i}{\partial x} - \frac{\partial \psi_{i-1}}{\partial x}\right)\Bigg|_{x=x_i} = 2(\Theta|_{x=x_i} - f_i(\tau)), \ i=2, 3, \tag{20}$$

$$\frac{\partial \psi_3}{\partial x}\Bigg|_{x=x_4} = 0. \tag{21}$$

The quantities $\partial J/\partial \lambda$ and $\partial J/\partial \mathbf{Q}$ are expressed in terms of the solution of the conjugate problem as follows:

$$\frac{\partial J}{\partial \lambda_n} = \sum_{i=1}^{3} \int_0^{\tau_m} \int_{x_i}^{x_{i+1}} \psi_i \frac{\partial}{\partial x}\left(\varphi_n(\Theta)\frac{\partial \Theta}{\partial x}\right) dx d\tau, \quad n=\overline{0, N};$$

$$\frac{\partial J}{\partial Q_n} = \sum_{i=1}^{3} \int_0^{\tau_m} \int_{x_i}^{x_{i+1}} \psi_i \varphi_n(\Theta)\, dx d\tau, \quad n=\overline{0, N}.$$

In the iterative descent of (15), (16) we used the linear vector estimate of the quantities $\beta_{\lambda k}$ and β_{Qk} [11].

The purpose of the second problem is the determination of $q(\tau)$ from the solution of the one-dimensional IPHE in the region $(0, x_1) \times (0, \tau_m)$ with known temperature and thermal flux density at the boundary [1], which is calculated from (8)-(13) from the reconstructed values of λ and Q. The point here is that the quantities $\lambda(T)$ and $Q(T)$ in the thermal conductivity equation were reconstructed only for $T \le T_{max}$, while in the region $x < x_1$ the temperature can exceed the value T_{max}. The reconstructed function $q(\tau)$ will then correspond to the real one for $\tau < \tau_0$, where τ_0 satisfies the condition $T(0, \tau_0) = T_{max}$.

At times $\tau > \tau_0$ due to the uniqueness theorem the reconstructed value of $q(\tau)$ may not have anything in common with the real solution.

For numerical realization of the algorithm the boundary value problems were approximated by algebraic equations with an implicit scheme [12] on a grid $n_\tau \times n_x = 50 \times 40$. The

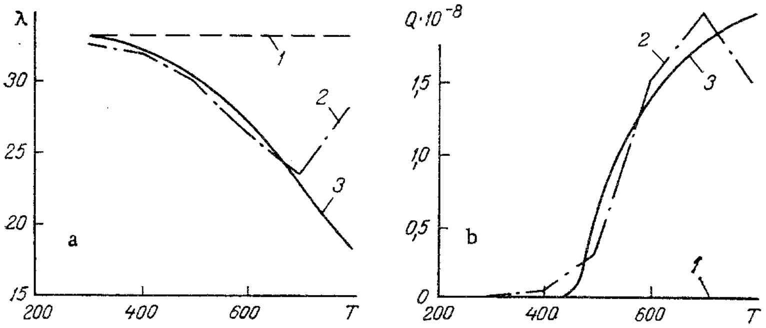

Fig. 2. Result of reconstructing the thermal conductivity coefficient (a) and the bulk heat release (b): 1) initial approximation; 2) reconstructed solution (10th interation); 3) real solution Q, W/m^3; T, K; λ, W/(m•K).

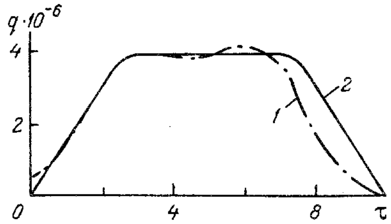

Fig. 3. Result of recovering the thermal flux density at zeroth initial approximation: 1) reconstructed solution (15th interaction); 2) model thermal flux density q, Вт/м^2; τ, с

computation time of a single iteration for the IPHE on an ES-1022 computer was 5 min, and the iteration was stopped when the adhesion condition was approximated within 0.5%. The input values $f_i(\tau)$, i = 1, 2, 3, were selected from the solution of the direct problem of thermal conductivity (1)-(3), (5). The following parameter values were taken: $\tau_m=10$sec, $b=10^{-2}$ m, $x_1=10^{-3}$ m, $x_2=4\cdot10^{-3}$ m, $x_3=8\cdot10^{-3}$ m, $T_0=300$ K, $c(T)=(2{,}64\cdot10^6+1{,}2+10^3\,T)$ J/(m^3•K). The number of N of approximation sites of the functions λ and Q were taken equal to 5, with T_{min} = 300°K, T_{max} = 792°K.

The solution results of the IPHE model are shown in Figs. 2 and 3. It is seen that upon approaching high temperature values the quality of recovering λ and Q worsens. Besides, as is seen from Fig. 3, the reconstructed and the real thermal flux density differ substantially from each other at times near τ_m, which is a direct consequence of the uniqueness theorem.

NOTATION

T, Θ, temperature fields; τ, current time; τ_m, observation time; x, spatial coordinate; c, specific heat; λ, thermal conductivity coefficient; Q, bulk heat release; q, thermal flux density; D_{α_2}, uniqueness region of the IPHE solution; T_{min} and T_{max}, minimal and maximal inlet temperature values; β_λ, β_Q, descent steps on iterations; ψ_i, conjugated function; and γ_1, γ_2, positive constants.

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STUDY OF THE UNSTEADY TEMPERATURE FIELD IN A SPHERICAL BODY USING CHEBYSHEV—LAGUERRE POLYNOMIALS

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We discuss a method of solving axisymmetric boundary-value problems for the parabolic heat equation in spherical coordinates based on the use of Chebyshev—Laguerre polynomials.

The unsteady heat conduction of a spherical body subject to nonuniform axisymmetric heating of its surface reduces to the solution of a boundary-value problem for the parabolic heat equation. The Laplace transform in time leads to significant computational difficulties in this case. We discuss a new method of finding the unsteady temperature field in a spherical body subject to local heating. The method is based on the use of Chebyshev—Laguerre polynomials [1].

1. Consider a hollow sphere and define spherical coordinates (r, θ, φ) in the usual way. The outer and inner surfaces of the sphere are subject to heat exchange according to Newton's law into media with temperatures $T_c^{\pm}(\theta, F)$, respectively.

The temperature field $T(\gamma, \theta, F)$ inside the sphere is found by solving the following axisymmetric mixed initial-value—boundary-value problem:

$$\frac{1}{(1+\varepsilon\gamma)^2}\frac{\partial}{\partial\gamma}\left[(1+\varepsilon\gamma)^2\frac{\partial T}{\partial\gamma}\right]+\frac{\varepsilon^2}{(1+\varepsilon\gamma)^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=\frac{\partial T}{\partial F},\quad -1\leqslant\gamma\leqslant 1,\quad 0\leqslant\theta\leqslant\pi, \tag{1}$$

$$\left(\frac{\partial T}{\partial\gamma}\right)^{\pm}\pm \mathrm{Bi}^{\pm}[T^{\pm}-T_c^{\pm}(F,\ \theta)]=0,\qquad \gamma=\pm 1, \tag{2}$$

$$T(\gamma,\ \theta,\ F)=0,\qquad F\leqslant 0, \tag{3}$$

where $r = R(1 + \varepsilon\gamma)$ is the radius of the sphere, $\varepsilon = h/R$; $F = a\tau/h^2$ is the Fourier number; $\mathrm{Bi}^{\pm} = \alpha^{\pm}h/\lambda_t$ are the Biot numbers on the surfaces $\gamma = \pm 1$.

The integral formula

$$T_{nm}(\gamma)=\left(m+\frac{1}{2}\right)\int_0^{\infty}\exp(-\lambda F)\,L_n(\lambda F)\left[\int_0^{\pi}T(\gamma,\ \theta,\ F)\,P_m(\cos\theta)\sin\theta d\theta\right]dF,\quad n,\quad m=\overline{0,\ \infty}, \tag{4}$$

defines a double integral transform of the function $T(\gamma, \theta, F)$, where $L_n(\lambda F)$ are the orthogonal Chebyshev—Laguerre polynomials; $P_m(\cos\theta)$ are the orthogonal Legendre polynomials [2]; λ is a positive parameter which we call the regularization parameter.

I. Franko State University, Lvov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 52, No. 5, pp. 844-851, May, 1987. Original article submitted February 5, 1987.

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